ELSEVIER

Journal of Geometry and Physics 29 (1999) 177–197

JOURNAL OF GEOMETRY AND PHYSICS

The b-completion of the Friedmann space ★

A.M. Amores [a,*], M. Gutiérrez [b]

[a] Departmento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain

[b] Departamento de Algebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain

Received 9 June 1998; received in revised form 7 July 1998

Abstract

We study the b-completion of the three Friedmann models of the Universe, having as models for 3-space the sphere, the Euclidean space or the hyperbolic space. We show that in the first case there is just one singularity, having the full completion as only neighborhood. In the other two cases there is one essential singularity, which is the limit of all past causal geodesics; again, it has a single neighborhood. This extends results by Bosshard [On the b-boundary of the closed Friendmann Model, Commun. Math. Phys. 46 (1976) 263–268] and Johnson [The bundle boundary in some special cases, J. Math. Phys. 18 (5) (1977) 898–902] on the closed Friedmann model. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Differential geometry; Lorentzian geometry; General relativity

1991 MSC: 53C50; 83C75

Keywords: Singularities; b-Completion; Friedmann space

1. Introduction

We briefly recall Schmidt's definition of the *b-completion* of a connected-oriented and time-oriented spacetime (see [2,3,8] for general results). Let $\pi : P \to M$ be the subbundle of the frame bundle given by the orthonormal positive frames $(u_0, u_1, \ldots, u_n)$, $\dim(M) = n + 1$, u_0 being timelike future. The structure group is $\mathbb{L}$, the group of Lorentz transformations of $\mathbb{R}^{n+1}$ preserving the natural orientation and time orientation. Let ϕ and ω be the fundamental form and the connection form (of the Levi-Civita connection), defined

★ The first author denotes this work to the memory of his father.

* Corresponding author.

PII S0393-0440(98)00030-2

on P with values in $\mathbb{R}^{n+1}$ and $gl(n+1;\mathbb{R})$, respectively. We have on P a Riemann metric determined by $G(X,X) = \|\phi(X)\|^2 + \|\omega(X)\|^2$, where $\|\cdot\|$ is either the Euclidean norm on $\mathbb{R}^{n+1}$ or the norm $\|a\|^2 = tr(aa^T)$ in $gl(n+1;\mathbb{R})$. Let d be the distance on P associated to G and let $(\hat{P},\hat{d})$ be the Cauchy completion of the metric space (P,d). The right action of $\mathbb{L}$ on P can be extended to $\hat{P}$, and the quotient space $\hat{M} = \hat{P}/G$ is defined as the b-completion of (M,g). The *set of singularities* or *b-boundary* is $\hat{M} - M$. Since the distance $\hat{d}$ on $\hat{P}$ and the projection $\hat{\pi} : \hat{P} \to \hat{M}$ extend d and π, we will often just write d and π.

A b-incomplete curve $c : [0,a) \to M$ is a smooth curve without limit as $t \to a$, having a finite length horizontal lift C for the metric G. It follows that C has a limit $p \in \hat{P}$ and that $\pi(p) \in \hat{M} - M$. We say then that c *induces* $\pi(p)$. It can be proved [4, Proposition 8.3.1] that any singularity can be obtained in this way.

Consider the following spacetime (M,g) where:
- $M = I \times N$, $I = (0,l)$, $l \leq \infty$, $N = \mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$ (hyperbolic space).
- h is the standard Riemann metric on N.
- $g = -\mathrm{d}t^2 + f^2 h$, where $f : (0,l) \to \mathbb{R}$ is the maximal solution of $(\dot{f}^2 + C)f = 2k$ (*Friedman equation*), such that $\lim_{t\to 0} f(t) = 0$ for the following choices of C: $C = 0$ if $N = \mathbb{R}^3$; $C = 1$ if $N = \mathbb{S}^3$; $C = -1$ if $N = \mathbb{H}^3$. Always $k > 0$.

The three spacetimes just defined are the Friedmann spaces, the closed one corresponding to $N = \mathbb{S}^3$. The *Friedmann surfaces* are the two-dimensional analogs taking $N = \mathbb{R}$ or $\mathbb{S}^1$ with the same f. If $c(t) = (x(t), y(t))$ is a b-incomplete curve such that $x(t_n) \to 0$ for some $t_n \to a = \sup(\mathrm{dom}(c))$, we say that the singularity induced is a *(past) essential singularity*. Since the function f in the closed space is defined on $(0, 2\pi k)$ and is symmetric in the sense that $f(\pi k - t) = f(\pi k + t)$, we could define in this case a *future essential singularity*, changing to the condition $x(t_n) \to 2\pi k$ for some $t_n \to a$.

The following results are known ([1,5]):
- The closed Friedmann surface has just one essential (past) singularity, whose only neighborhood contains the space.
- There is at least one essential singularity in the closed Friedmann space such that its neighborhoods contain the space.
- Some past and future essential singularities in the Friedmann closed space actually coincide.

We improve these results as follows
- The closed Friedmann space has just one singularity (of any kind, essential or not) with the whole completion as only neighborhood. (It was conjectured in [5] that there was only one past essential singularity.)
- The nonclosed Friedman spaces have a unique past essential singularity; its neighborhoods contain the space.
- In nonclosed Friedman spaces, causal future maximal geodesics have horizontal lifts with infinite length. Therefore, they are not b-incomplete and do not seem to induce (even in a loose sense) any "future singularity".

When the papers of Bosshard and Johnson appeared, the trivial convergence of any curve to a singularity, plus the coincidence of Big Bang and Big Crunch singularities were considered undesirable features of the b-completion. Still, there could exist some undetected (i.e. different from the essential) singularities in the closed model, not so badly behaved. Also, it could happen that the situation improved in the nonclosed models. Our paper shows that this is not the case, and that the drawbacks of the definition of b-completion appear in all examples of the Friedmann space.

The paper is structured as follows. In Section 2 we study some properties of null geodesics; in Section 3 we give formulas for parallel fields along certain curves, allowing us to construct horizontal lifts and compute its length in Section 4. We work up to this point in a spacetime which is more general than the Friedmann spaces. In Section 5 we choose $N = \mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, and define, via a special global frame, a section σ of π, whose properties will give us information on the structure of $\hat{P} - P$. In Section 6 we work with the Friedmann models obtaining the main theorems. There are important ideas in the general structure of the proof which are traceable to the paper of Johnson.

2. Some properties of null geodesics

Let $I = (0, l)$ be an interval, $l \leq \infty$ and $f : I \to \mathbb{R}$ a strictly positive smooth map such that $f(x) \to 0$ and $\dot{f}(x) \to 0$. We denote by U the natural field $\partial/\partial t$. The function $\varphi = U \cdot f/f = \dot{f}/f$ will often appear in the paper. *We choose an oriented, complete and connected Riemann n-dimensional manifold* (N, h) *and consider the warped product metric on* $M = I \times N$ *with formula* $g = -dt \otimes dt + f^2 h$. (See [7] as a reference for warped products.)

The fields $U \in \mathcal{X}(I), X, Y \in \mathcal{X}(N)$ will be identified with fields in $\mathcal{X}(M)$. If D is the Levi-Civita connection of (N, h) we have the following formulas for the Levi-Civita connection ∇ of (M, g)

$$\nabla_U U = 0, \quad \nabla_U X = \nabla_X U = \varphi X, \quad \nabla_X Y = D_X y + \varphi g(X, Y) U.$$

We denote by $\mathcal{X}(\gamma)$ the space of fields along the curve γ. Let $c = (x, y)$ be a curve in M; therefore $x : J \to \mathbb{R}, y : J \to N$. A field V along c can be represented in the form $V = r(U \circ c) + K$, with $r : J \to \mathbb{R}$, $K \in \mathcal{X}(y)$. The covariant derivative of ∇V of V is given by

$$\begin{aligned} \nabla V = &(\dot{r} + (\varphi \circ x) g(\dot{y}, K))(U \circ c) \\ &+ (DK + (\varphi \circ x)(\dot{x} K + r \dot{y})). \end{aligned} \tag{2.1}$$

It is clear then that V is parallel if the following equations hold:

$$\dot{r} + (\varphi \circ x) g(\dot{y}, K) = 0, \tag{2.2a}$$

$$DK + (\varphi \circ x)(\dot{x} K + r \dot{y}) = 0. \tag{2.2b}$$

In particular, if $V = \dot{c}$, we have $r = \dot{x}$, $K = \dot{y}$. Hence, c is a geodesic if

$$\ddot{x} + (\varphi \circ x) g(\dot{y}, \dot{y}) = 0, \tag{2.3a}$$

$$D\dot{y} + (\varphi \circ x)(2\dot{x}\dot{y}) = 0. \tag{2.3b}$$

We are interested in null geodesics, which can be obtained by quadratures.

Theorem 2.1. *Let $c : [0, \tau) \to M$ be a null geodesic. Then,*

$$\dot{x}^2 = g(\dot{y}, \dot{y}), \tag{2.4a}$$
$$(f \circ x)\dot{x} = k(\text{constant}), \tag{2.4b}$$

and x is given by

$$kt = \int_{x(0)}^{x(t)} f(\xi)\,\mathrm{d}\xi. \tag{2.5}$$

Moreover, if we reparameterize y to y_1, say $y = y_1 \circ s$, with $\dot{s} > 0$, $\varepsilon = sign(\dot{x}) = \pm 1$ and $h(\dot{y}_1, \dot{y}_1) = 1$, then y_1 is a geodesic of (N, h) and s is given by

$$s(t) - s(0) = \int_{x(0)}^{x(t)} \frac{\varepsilon\,\mathrm{d}\xi}{f(\xi)}. \tag{2.6}$$

Proof. Clearly $\dot{c} = \dot{x}(U \circ c) + \dot{y}$, and it will be null if and only if (2.4a) holds. Substituting (2.4a) in (2.3a) we get, since $\varphi = \dot{f}/f$, that $(f \circ x)\ddot{x} + (\dot{f} \circ x)\dot{x}^2 = 0$; but this is just (2.4b). If we integrate (2.4b) we get

$$kt = \int_0^t (f \circ x(u))\dot{x}(u)\,\mathrm{d}u = \int_{x(0)}^{x(t)} f(\xi)\,\mathrm{d}\xi.$$

Since the derivative of $\int_0^x f(\xi)\,\mathrm{d}\xi$ is $f(x) > 0$, x is well determined by the integral. Condition (2.3b) implies that y is a pregeodesic for D; hence if reparametrized to y_1 it will give a geodesic of D. We have, using (2.4a),

$$\begin{aligned}\dot{x}^2 &= g(\dot{y}, \dot{y}) = \dot{s}^2 g(\dot{y}_1 \circ s, \dot{y}_1 \circ s)\\ &= \dot{s}^2 f^2(x) h(\dot{y}_1 \circ s, \dot{y}_1 \circ s) = \dot{s}^2 f^2(x).\end{aligned}$$

Now, since $(\dot{x}^2)^{1/2} = |\dot{x}| = \varepsilon\dot{x}$ and $\dot{s} f > 0$, we get (2.6) by integration. □

Theorem 2.2. *Let $c = (x, y) : [0, \tau) \to M$ be a maximal null geodesic. Then $x(t)$ con verges to one of the endpoints of I as $t \to \tau$. If $\dot{x}(0) = -1$,*

$$\tau = \frac{1}{f(x(0))} \int_0^{x(0)} f(\xi)\,\mathrm{d}\xi < \infty. \tag{2.7}$$

Proof. First of all, x has a limit (for $\dot{x}$ is never 0); suppose it were not 0 or l; then by (2.6), r would have a finite limit as $t \to \tau$. By completeness, y_1 is defined on $\mathbb{R}$, implying that y has a limit too. We have shown that c has a limit, contrdicting maximality. Formula (2.7) follows (2.4b) and (2.5) taking limit as $t \to \tau$. □

For technical reasons it is often better to consider null pregeodesics.

Theorem 2.3. *Let* $y : [0, \tau) \to N$ *such that* $D\dot{y} = 0, h(\dot{y}, \dot{y}) = 1$. *Then the curve in* M, $c(t) = (x(t), y(t))$ *is a null pregeodesic if* $\dot{x}^2 = (f \circ x)^2$; *in fact, if* $\dot{x} = \epsilon f \circ x$, $\epsilon = \pm 1$, *we have* $\nabla\dot{c} = 2\epsilon(\dot{f} \circ x)\dot{c}$ *and* $\ddot{x} = \epsilon(\dot{f} \circ x)\dot{x} = (f \circ x)(\dot{f} \circ x)$. *Moreover, if* c *is a reparameterization of a* maximal *null past geodesic then*

$$\tau = \int_0^{x(0)} \frac{d\xi}{f(\xi)} \leq \infty. \tag{2.8}$$

Proof. Since $\dot{c} = \dot{x}(U \circ c) + \dot{y}$, $g(\dot{c}, \dot{c}) = 0$ implies $\dot{x}^2 = (f \circ x)^2$; hence $\dot{x} = \epsilon f \circ x$ and $\ddot{x} = (f \circ x)(\dot{f} \circ x)$. Now, by (2.1) and (2.3a,b)

$$\begin{aligned}\nabla\dot{c} &= [\ddot{x} + (\varphi \circ x)(f \circ x)^2 h(\dot{y}, \dot{y})](U \circ c) \\ &\quad + [D\dot{y} + 2(\varphi \circ x)\dot{x}\dot{y}] \\ &= 2(f \circ x)(\dot{f} \circ x)(U \circ c) + 2\epsilon(\dot{f} \circ x)\dot{y} \\ &= 2\epsilon(\dot{f} \circ x)[\dot{x}(U \circ c) + \dot{y}] = 2\epsilon(\dot{f} \circ x)\dot{c}.\end{aligned}$$

Formula (2.8) follows by integration of $\dot{x}/f \circ x = -1$, taking limits as $t \to \tau$, since, by Theorem 2.2, $x(t) \to 0$ as $t \to \tau$. □

3. Parallel fields

We compute in the lemmas in this section the expression of the most general parallel field V along certain particular curves $c = (x, y) : J \to M$.

Lemma 3.1. *For fixed* $(a, b) \in M$ *define* $c(t) = (a - t, b)$, $t \in [0, a)$. *Then*

$$V(t) = X(U \circ c)(t) + \frac{f(a)}{f(a-t)}\xi,$$

where $X \in \mathbb{R}$ *and* $\xi \in T_bN$.

Proof. Just apply Eqs. (2.2a) and (2.2b) to $r = X$ (constant) and $K(t) = (f(a)/f(a-t))\xi$, which is a field along the constant curve $y = b$. □

Lemma 3.2. *Let* $c = (x, y) : J \to M$ *be a curve such that* y *is a D-geodesic in* N *with* $h(\dot{y}, \dot{y}) = 1$. *Then*

$$V = (X_0 \operatorname{ch}\theta + Y_0 \operatorname{sh}\theta)U \circ c + (X_0 \operatorname{sh}\theta + Y_0 \operatorname{ch}\theta)\frac{\dot{y}}{f \circ x} + \frac{P}{f \circ x},$$

where $X_0, Y_0 \in \mathbb{R}$, $P \in \mathcal{X}(y)$ *such that* $h(\dot{y}, P) = 0$ *and* $DP = 0$, *and* $\theta : J \to \mathbb{R}$ *is a solution of* $\dot{\theta} + \dot{f} \circ x = 0$.

Proof. Any $V \in \mathcal{X}(c)$ can be written as

$$V = X(U \circ c) + Y(\dot{y}/f \circ x) + \sum Z^i P_i$$

with $X, Y, Z^i : J \to \mathbb{R}$ and $P_i \in \mathcal{X}(y)$, $h(\dot{y}, P_i) = 0$, $DP_i = 0$, $i = 2, \dots, n$. Eqs. (2.2a) and (2.2b) are equivalent to $n + 1$ scalar equations

$$\dot{X} + (\dot{f} \circ x)Y = 0, \quad \dot{Y} + (\dot{f} \circ x)X = 0,$$
$$\dot{Z}^k + \varphi(x)\dot{x}Z^k = 0 \quad (2 \leq k \leq n),$$

which are easily solved, giving the formula for V. □

Two particular cases of Lemma 3.2 will appear soon. In the first case $x = a$ (constant); in the second one c is a null pregeodesic. We get as solution θ with initial condition $\theta(0) = 0$, depending on the case, the functions

$$\theta(t) = -\dot{f}(a)t, \quad \theta(t) = \log\left(\frac{\dot{x}(t)}{\dot{x}(0)}\right) = \log\left(\frac{f(x(t))}{f(x(0))}\right), \tag{3.1}$$

having used for c null past pregeodesic that $\dot{x} = -f \circ x$ (Theorem 2.3).

4. The length of certain lifts

Let $c : J \to M$ be a curve in M and $C : J \to P$ a lift of c to P; thus C is a moving frame $(C_1, \dots, C_m)$. We can then write

$$\dot{c} = \sum c^i C_i, \quad \nabla C_i = \sum f_i^j C_j, \quad c^i, f_i^j : J \to \mathbb{R}. \tag{4.1}$$

With the general notation is Section 1 we have formulas

$$\omega(\dot{C}) = (f_i^j), \quad \phi(\dot{C}) = (c^i), \quad G(\dot{C}, \dot{C}) = \sum (c^i)^2 + \sum (f_i^j)^2. \tag{4.2}$$

We use them in the lemma below to compute the lengths of certain lifts C of curves c of one of the special forms considered in Section 3. *$L(\gamma)$ will always denote the length of a curve γ.*

Lemma 4.1. *Fix $(a, b) \in M$ and let $c(t) = (a - t, b)$, $t \in [0, a)$. Choose $\xi_1, \dots, \xi_n$ in T_bN such that $h(\xi_i, \xi_j) = f(a)^{-2}\delta_{ij}$; thus $(U(a, b), \xi_1, \dots, \xi_n)$ is a g-orthonormal basis at (a, b). Define $C : [0, a) \to P$ by*

$$C_0 = U \circ c, \qquad C_i(t) = \frac{f(a)}{f(a - t)}\xi_i \quad (i > 0).$$

Then C is a horizontal lift of C, $G(\dot{C}, \dot{C}) = 1$ and $L(C) = a$.

Proof. By Lemma 3.1 all C_k are parallel. With the notation in (4.1), $\dot{c} = -U \circ c$, hence $c^0 = -1$ and $c^i = 0$ for $i > 0$ so we just apply (4.2). □

Lemma 4.2. *Consider $c:(-\tau,\tau)\to M$, $c(t)=(a,y(t))$, where y is a D-geodesic in N such that $h(\dot y,\dot y)=1$. Let C be given by*

$$C_0(t)=\mathrm{ch}(-\dot f(a)t)U(c(t))+\mathrm{sh}\,(-\dot f(a)t)(\dot y)(t)/f(a)),$$
$$C_1(t)=\mathrm{sh}(-\dot f(a)t)U(c(t))+\mathrm{ch}\,(-\dot f(a)t)(\dot y(t)/f(a)),$$
$$C_i=(1/f(a))P_i\quad \text{for } i\geq 2,$$

where $P_i\in\mathcal{X}(y)$, $g(\dot y,P_i)=0$, $DP_i=0$, $h(P_i(0),P_j(0))=\delta_{ij}$. Then, C is a horizontal lift of c and we have

$$G(\dot C,\dot C)(t)=f(a)^2\,\mathrm{ch}(2\dot f(a)t)\leq 2f(a)^2\,\mathrm{ch}^2(\dot f(a)t),$$
$$L(C)\leq\frac{2\sqrt{2}}{\varphi(a)}sh(\dot f(a)\tau).$$

Proof. All C_i are parallel by Lemma 3.2. With an obvious matrix notation,

$$\begin{pmatrix}C_0\\ C_1\end{pmatrix}=\begin{pmatrix}\mathrm{ch}\,(-\dot f(a)t) & \mathrm{sh}(-\dot f(a)t)\\ \mathrm{sh}(-\dot f(a)t) & \mathrm{ch}\,(-\dot f(a)t)\end{pmatrix}\begin{pmatrix}U\circ c\\ \dot y/f(a)\end{pmatrix},$$
$$\begin{pmatrix}U\circ c\\ \dot y/f(a)\end{pmatrix}=\begin{pmatrix}\mathrm{ch}\,(\dot f(a)t) & \mathrm{sh}(\dot f(a)t)\\ \mathrm{sh}(\dot f(a)t) & \mathrm{ch}\,(\dot f(a)t)\end{pmatrix}\begin{pmatrix}C_0\\ C_1\end{pmatrix}.$$

In particular, since $\dot c=\dot y$, we get using notation (4.1) that all $f_i^j=0$ and

$$c^0(t)=f(a)\mathrm{sh}(\dot f(a)t),\quad c^1(t)=f(a)\,\mathrm{ch}\,(\dot f(a)t),$$
$$c^i=0\quad \text{for } i>1.$$

The lemma is now immediate. □

Lemma 4.3. *Let $c=(x,y):[0,\tau)\to M$ be a null maximal past pregeodesic such that $D\dot y=0$ and $h(\dot y,\dot y)=1$. Define a lift $C=(C_0,\dots,C_n)$ of c by*

$$C_0=\mathrm{ch}\,\theta(U\circ c)+sh\,\theta\frac{\dot y}{f\circ x},\quad C_1=sh\,\theta(U\circ c)+\mathrm{ch}\,\theta\frac{\dot y}{f\circ x},$$
$$C_i=\frac{1}{f\circ x}P_i,$$

where $P_i\in\mathcal{X}(y)$, $DP_i=0$, $h(\dot y,P)=0$, $h(P_i,P_j)=\delta_{ij}$ and $\theta:[0,\tau)\to\mathbb{R}$ is given by $\exp(\theta)=\dot x/\dot x(0)$. *Then C is a horizontal lift of c and*

$$L(C)=\sqrt{2}\int_0^{x(0)}\frac{f(\xi)\,\mathrm{d}\xi}{f(x(0))}.$$

Proof. All C_k are parallel by Lemma 3.2. We get as in the proof of Lemma 4.2,

$$U\circ c=\mathrm{ch}\,\theta C_0-\mathrm{sh}\,\theta C_1,\quad \dot y/(f\circ c)=-\mathrm{sh}\,\theta C_0+\mathrm{ch}\,\theta C_1,$$
$$\dot c=\dot x(U\circ c)+\dot y=\dot x\exp\theta(C_0-C_1)=(\dot x^2/\dot x(0))(C_0+C_1).$$

Using (4.2) and $\lim_{t\to\tau} x(t) = 0$ (see Theorem 2.2) we arrive at

$$L(C) = \sqrt{2}\int_0^{\tau} \frac{\dot{x}^2(t)\,\mathrm{d}t}{-\dot{x}(0)}$$
$$= \sqrt{2}\int_0^{\tau} \frac{-f(x(t))\dot{x}(t)}{f(x(0))}\,\mathrm{d}t = \sqrt{2}\int_0^{x(0)} \frac{f(\xi)\,\mathrm{d}\xi}{f(x(0))}. \qquad \square$$

Suppose from now onwards that there is a fixed global orthonormal positive frame on N, denoted by $F_1, \ldots, F_n$. Since $h(F_i, F_j) = \delta_{ij}$, the fields E_i on M

$$E_0 = U, \qquad E_i(x, y) = f(x)^{-1}F_i(y) \quad (1 \le i \le n) \tag{4.3}$$

are a g-orthonomal frame on M. This frame gives a section σ of $\pi : P \to M$ whose properties will be essential in this paper.

Lemma 4.4. *Let $c = (x, y) : [0, \tau] \to M$ be a curve. If $\dot{y} = \sum y^k F_k$ we have*

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}t}(\sigma \circ c)\right) = (\dot{x}, (f \circ x)y^1, \ldots, (f \circ x)y^n),$$
$$\omega\left(\frac{\mathrm{d}}{\mathrm{d}t}(\sigma \circ c)\right) = \begin{pmatrix} 0 & (\dot{f} \circ x)Y^T \\ (\dot{f} \circ x)Y & H \end{pmatrix}, \text{ and}$$
$$L(\sigma \circ c) = \int_0^{\tau} (\dot{x}^2 + (f \circ x)^2\|Y\|^2 + 2(\dot{f} \circ x)^2\|Y\|^2 + \|H\|^2)^{1/2}\,\mathrm{d}t,$$

where $Y = (y^1, \ldots, y^n)$ is identified to a column vector, H is the $n \times n$ matrix $H_{ji} = h(D(F_i \circ y), (F_j \circ y))$ and $\|H\|^2 = \sum (H_{ji})^2$.

Proof. We apply (4.2) once more to $C = \sigma \circ c$. We have

$$\dot{c} = \dot{x}(U \circ c) + \dot{y} = \dot{x}(E_0 \circ c) + f(x)\sum y^k(E_k \circ c),$$

and the first formula follows. Now we apply (2.1) choosing either $r = 1$ and $K = 0$ or $r = 0$ and $K = E_i \circ c = (f \circ x)^{-1}(F_i \circ y)$. We may compute then

$$\nabla(U \circ c) = \varphi(x)\dot{y} = \sum \dot{f}(x)y^k E_k \text{ and}$$
$$\begin{aligned}\nabla(E_i \circ c) &= \varphi(x)f(x)^2 h(\dot{y}, E_i \circ c)(U \circ c) \\ &\quad + D(E_i \circ c) + \varphi(x)\dot{x}(E_i \circ c) \\ &= \dot{f}(x)y^i(U \circ c) + \frac{1}{f(x)}D(F_i \circ y) \\ &\quad - \frac{\dot{f}(x)\dot{x}}{f(x)^2}(F_i \circ y) + \varphi(x)\dot{x}(E_i \circ c).\end{aligned}$$

The last two summands cancel and the second and third formulas are clear. $\square$

5. Constructions for special N in dimension 3

Denote by $\mathbb{R}^3$ and $\mathbb{R}^4$ the Euclidean space or the Minkowski space; (e_1, e_2, e_3) and (e_0, e_1, e_2, e_3) will be the standard basis with $e_0 = (1, 0, 0, 0)$, assuming $\langle e_0, e_0\rangle = -1$. Let N be either $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ with its natural Riemann metric h. Here the sphere $\mathbb{S}^3$ is considered as the group of unit quaternions and $\mathbb{H}^3$ is the hyperbolic space. We describe the model of $\mathbb{H}^3$ to be considered. Let $\langle .,.\rangle$ be the Lorentz product of $\mathbb{R}^4$; then $\mathbb{H}^3 = \{x \in \mathbb{R}^4 | \langle x, y\rangle = -1, x^0 > 0\}$ and the metric h on $\mathbb{H}^3$ is the one inherited as a submanifold of Minkowski space.

In each case, N has a distinguished point $\circ = 0, 1, e_0$ depending on $N = \mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$. We construct a special orthonormal frame F_i on N. If $N = \mathbb{R}^3$ we pick up as F_i the parallel frame field such that $F_i(0) = e_i$. In the case $N = \mathbb{S}^3$, we know that the Lie algebra of the unit quaternion group can be naturally identified to $\mathbb{R}^3$ (the tangent space at 1 is the set of pure quaternions); we choose as F_i the invariant fields on $\mathbb{S}^3$ induced by the e_i. If $N = \mathbb{H}^3$ we choose as F_i the geodesic frame obtained from the tangent vectors e_1, e_2, e_3 at e_0. More precisely, given $x \in \mathbb{H}^3$, $x \neq e_0$, we write it (uniquely) as

$$x = \mathrm{ch}\,(\theta)e_0 + sh(\theta)u, \quad \theta > 0,$$
$$\langle u, e_0\rangle = 0, \quad \langle u, u\rangle = 1,$$

and we consider the geodesic joining e_0 to x,

$$c_x : [0, 1] \to \mathbb{H}^3, \qquad c_x(t) = ch(t\theta)e_0 + sh(t\theta)u.$$

Then, we have three parallel fields V_i along c_x given by the initial condition $V_i(0) = e_i$, and we define $F_i(x) = V_i(1)$. If $y : \mathbb{R} \to \mathbb{H}^3$ is the geodesic

$$y(t) = \mathrm{ch}\,(t)e_0 + \mathrm{sh}(t)e_1,$$

one may check that $F_1(y(t)) = \dot{y}(t)$, $F_j(y(t)) = e_j$ for $j = 2, 3$.

Lemma 5.1. *Let $y : \mathbb{R} \to N$ be the geodesic determined by the conditions $y(0) = o$ and $\dot{y}(0) = e_1$. Then, if $\vee$ is the vector product in $\mathbb{R}^3$.*

$$D(F_i \circ y) = 0 \quad \text{for } i = 1, 2, 3 \text{ and } N = \mathbb{R}^3, \mathbb{H}^3;$$
$$D(F_1 \circ y) = 0, \qquad D(F_i \circ y) = (F_1 \vee F_i) \circ y$$
$$\text{for } i = 2, 3 \text{ and } N = \mathbb{S}^3.$$

Proof. It is clear for $N = \mathbb{R}^3$ or $\mathbb{H}^3$ since the F_i are parallel along any curve ($N = \mathbb{R}^3$) or are parallel along geodesics starting at $e_0 (N = \mathbb{H}^3)$. The analogous statement for $N = \mathbb{S}^3$ follows from the formula $D_X Y = X \vee Y$, X, Y invariant fields, since $\dot{y} = F_1 \circ y$. □

Lemma 5.2. *For each $b \in N$ there is an isometry φ of (N, h) such that*

$$\varphi(o) = b \text{ and } T\varphi(F_i(o)) = F_i(b) \quad (1 \leq i \leq 3).$$

Proof. For $N = \mathbb{R}^3$ or $\mathbb{S}^3$ choose $\varphi(x) = x + b(N = \mathbb{R}^3)$ or $\varphi(x) = bx(N = \mathbb{S}^3)$. If $N = \mathbb{H}^3$ write $b = ch(\theta)e_0 + sh(\theta)u$ for $u \in e_0^\perp$, $|u| = 1$. Define $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ as a linear map given by the conditions:

$$\varphi(x) = x \quad \text{if } \langle x, e_0\rangle = \langle x, u\rangle = 0,$$
$$\varphi(e_0) = \mathrm{ch}\,(\theta)e_0 + \mathrm{sh}\,(\theta)u,$$
$$\varphi(u) = \mathrm{sh}\,(\theta)e_0 + \mathrm{ch}\,(\theta)u.$$

We denote by the same symbol φ its restriction to an isometry of $\mathbb{H}^3$. Clearly, $\gamma(t) = \mathrm{ch}\,(t\theta)e_0 + \mathrm{sh}\,(t\theta)u$ is a geodesic joining e_0 and b and the fields $F_i \circ \gamma$ are parallel. Consider the linear maps

$$\mathrm{Par},\ \mathrm{Tan} : T_{e_0}\mathbb{H}^3 \to T_b\mathbb{H}^3$$

given by the parallel transport along γ and the tangent of φ. We only need show that Par=Tan, clecking it on the vector $\dot\gamma(0)$ and v such that $< v, \dot\gamma(0) >= 0$. First, $\mathrm{Tan}(\dot\gamma(0)) = \dot\gamma(1)$ by direct check, and $\dot\gamma(1) = \mathrm{Par}(\dot\gamma(0))$ because γ is a geodesic. On the other hand, v being tangent at e_0, $< v, e_0 >= 0$; hence, $< v, u >=< v, e_0 >= 0$. A field V along γ is parallel if and only if the condition $0 = \dot V+ < \gamma, V > \gamma$ holds. We check that, for our particular v, $V(t) = v$ (constant) is parallel. It follows now that, if D is the ordinary derivative,

$$\mathrm{Par}(v) = V(1) = v = \varphi(v) = D\varphi(e_0)(v) = \mathrm{Tan}(v). \qquad \square$$

We study the section α of P given by the fields E_i defined by formula (4.3), after our choice of the frame F_i. The technical properties of σ and an extension of σ to be defined will be essential to understand $\hat P$.

We use Lemma (4.4) to compute the length of $\sigma \circ c$ when c has $x = a$ (constant) and y is a unit-speed geodesic of N starting at o. In all cases $\|Y\| = 1$ and, if $N = \mathbb{R}^3$ or $\mathbb{H}^3$, $H = 0$. If $N = \mathbb{S}^3$ we will use that for invariant X, Y, we have $D_X Y = X \vee Y$, where $\vee$ is the vector product. It is now easy to get that H is the matrix of the endomorphism $Z \to Y \vee Z$. Lemma 4.4. gives

$$L(\sigma \circ c) = \tau(f(a)^2 + 2(\dot f(a)^2)^{1/2} \quad (N = \mathbb{R}^3 \text{ or } \mathbb{H}^3), \tag{5.1}$$
$$L(\sigma \circ c) = \tau(f(a)^2 + 2\dot f(a)^2 + 2)^{1/2} \quad (N = \mathbb{S}^3). \tag{5.2}$$

We extend the section $\sigma : M \to P$ to $[0, l) \times N$ as follows. Given $b \in B$, consider the curve $c(t) = (a - t, b)$ defined on $[0, a)$. By Lemma 4.1, $\sigma \circ c$ is a horizontal lift of c with length a, as we see by choosing $\xi_i = f(a)^{-1}F_i(b) = E_i(a, b)$. Then $\sigma \circ c$ converges to a point in $\hat P$ that will be, by definition, $\sigma(0, b)$. The definition is correct because it does not depend on a.

Theorem 5.1. *The extension $\sigma : [0, l) \times N \to \hat P$ is continuous.*

Proof. Only the continuity at $(0, b)$ needs proof. Given $\epsilon > 0$ we will choose δ and a neighborhood B of b in N such that if $x \leq \delta$ and $y \in B$, then $d(\sigma(0, b), \sigma(x, y)) \leq 3\epsilon$.

We have just pointed out that the curve $\sigma(a-t, b)$ joins $\sigma(a, b)$ to $\sigma(0, b)$ and has length a. Therefore

$$\mathrm{d}(\sigma(a,b), \sigma(0,b)) \leq a \quad \text{for any } b \in N. \tag{5.3}$$

We choose as B a D-geodesic ball centered at b of radius r such that

$$r(f(\epsilon)^2 + 2(\dot{f}(\epsilon)^2 + 2)^{1/2} \leq \epsilon.$$

Any $y \in B$ can be written as $\gamma(\tau)$, where $\gamma : [0, \tau] \to B$ is a D-geodesic such that $\gamma(0) = b$ and $h(\dot{\gamma}, \dot{\gamma}) = 1$. Since γ is unit-speed, $\tau < r$. We apply either (5.1) or (5.2) to the curve $c(t) = (\epsilon, \gamma(t))$ and we get

$$d(\sigma(\epsilon, b), \sigma(\epsilon, y)) \leq L(\sigma \circ c) \leq \tau(f(\epsilon)^2 + 2\dot{f}(\epsilon)^2 + 2)^{1/2} \leq \epsilon.$$

With this and (5.3), it is clear that taking $\epsilon = \delta$ we have, by the triangle inequality, $d(\sigma(0,b), \sigma(x,y)) \leq 3\epsilon$ for $x \in [0, \delta)$ and $y \in B$. □

In Minkowski space $\mathbb{R}^{n+1}$ choose a positive orthonormal basis $e_0, e_1, \ldots, e_n$ such that $e_0 = (1, 0, \ldots, 0)$. For $i \in \{1, \ldots, n\}$ and $r \in \mathbb{R}$, define a linear map by

$$\beta(e_0) = ch(r)e_0 + sh(r)e_i,$$
$$\beta(e_i) = sh(r)e_0 + ch(r)e_i, \quad \beta(e_j)e_j \ \ (j \neq i).$$

We say that β is a boost (or i-boost, to be more precise) with parameter r. Boosts are in $\mathbb{L}$, the structure group of P. Analogously, choose $i, j \in \{1, \ldots, n\}$ and $r \in \mathbb{R}$, and define elementary rotations by

$$\rho(e_i) = \cos(r)e_i + \sin(r)e_j,$$
$$\rho(e_j) = -\sin(r)e_i + \cos(r)e_j,$$
$$\rho(e_k) = e_k$$

for $k \neq i, j$. These rotations are also in $\mathbb{L}$. We are interested in $\mathbb{R}^4$. For the time being we will deal only with 1-boosts fixing pointwise span$\{e_2, e_3\}$ and rotations fixing pointwise span$\{e_0, e_1\}$. Since they are then determined just by the parameter r, we will use the notation $\beta(r)$ and $\rho(r)$ for them. The letters β and ρ will represent boosts and rotations in many formulas below.

Consider $c : \mathbb{R} \to M$, $c(t) = (a, y(t))$, where y is the geodesic in N determined by $y(0) = o$ and $\dot{y}(0) = e_1$. This curve has a lift $c : \mathbb{R} \to P$ given by

$$\begin{aligned} C_0(t) &= \mathrm{ch}\,(-\dot{f}(a)t)U(c(t)) + \mathrm{sh}(-\dot{f}(a)t)(\dot{y}(t)/f(a)), \\ C_1(t) &= \mathrm{sh}(-\dot{f}(a)t)U(c(t)) + \mathrm{ch}\,(-\dot{f}(a)t)(\dot{y}(t)/f(a)), \\ C_i(t) &= E_i(c(t)), \quad i = 2, 3. \end{aligned} \tag{5.4}$$

Since, $\dot{y} = F_1 \circ y$, $\dot{y}/f(a) = E_1 \circ c$ can be substituted in (5.4) and it is clear that $C(t) = \sigma(c(t)) \cdot \beta(-\dot{f}(a)t)$. If $N = \mathbb{R}^3, \mathbb{H}^3$, the fields $P_i = F_i \circ y$ are parallel along y. Therefore, C is in this case just the horizontal curve in Lemma 4.2 and we have the bound

$$L\left(C|_{[-\tau,\tau]}\right) \leq \frac{2\sqrt{2}}{\varphi(a)} sh(\dot{f}(a)\tau). \tag{5.5}$$

However, if $N = \mathbb{S}^3$, C is not horizontal. We compute $G(\dot{C}, \dot{C})$ with formulas (4.2). Indeed, with the ideas in the proof of Lemma 4.2 we get

$$c^0(t) = f(a)sh(\dot{f}(a)t), \quad c^1(t) = f(a)ch(\dot{f}(a)t),$$
$$c^i = 0 \text{ for } i > 1.$$

Also C_0 and C_1 are ∇-parallel by Lemma 4.2. Finally, by formula (2.1) we have $\nabla C_2 = C_3$, $\nabla C_3 = -C_2$. We have shown that all f_i^j are 0 except $f_2^3 = -f_3^2 = 1$. Therefore

$$\begin{aligned} &G(\dot{C}, \dot{C})(t) = f(a)^2 ch(2\dot{f}(a)t) + 2 \le 2f(a)^2 ch^2(\dot{f}(a)t) + 2, \\ &L\left(C|_{[-\tau,\tau]}\right) \le \frac{2\sqrt{2}}{\varphi(a)}\mathrm{sh}(\dot{f}(a)\tau) + 2\sqrt{2}\tau. \end{aligned} \tag{5.6}$$

Theorem 5.2. *If β is an i-boost, $\sigma(0, b) \cdot \beta = \sigma(0, b)$ for all $b \in N$.*

Proof. We prove it for 1-boosts, the other cases being analogous.

We claim it is enough to prove the theorem for $b = o$. Choose an isometry φ of N as in Lemma 5.2 and denote by Φ the isometry of M given by $\Phi(x, y) = (x, \varphi(y))$. Clearly Φ has a lift $\Phi_\#$ to the frame bundle P, which is an isometry of the metric G. This implies that $\Phi_\#$ can be extended to the completion $\hat{P}$ of P. It can also be checked that the left action of the isometry group of M on P or $\hat{P}$ commutes with the right action of the structure group $\mathbb{L}$, therefore $\Phi_\# \cdot u \cdot \lambda$ for $u \in P$ and $\lambda \in \mathbb{L}$ is unambigous. We have then

$$\begin{aligned} \sigma(0, b) \cdot \beta &= \sigma(0, \varphi(o)) \cdot \beta = (\Phi_\# \cdot \sigma(0, o)) \cdot \beta = \Phi_\# \cdot (\sigma(0, o) \cdot \beta) \\ &= \Phi_\# \cdot \sigma(0, o) = \sigma(0, \varphi(o)) = \sigma(0, b). \end{aligned}$$

We deal now with the case $b = o$. Suppose the 1-boost is $\beta = \beta(r)$. Consider for each $a \in I$ the curve C, depending on a, given in (5.4); it will be denoted by C_a. Write $r = -\dot{f}(a)T(a)$ and $\tau(a) = |T(a)|$. Clearly, C_a joins $\sigma(a, b)$ to $\sigma(a, y(T(a) \cdot \beta(r)$, and by (5.5) and (5.6),

$$\begin{aligned} \mathrm{d}(\sigma(a, b), \sigma(a, y(T(a)) \cdot \beta) &\le L\left(C|_{[-\tau(a),\tau(a)]}\right) \\ &\le \frac{2\sqrt{2}}{\varphi(a)} sh(|r|) + 2\sqrt{2}\tau(a). \end{aligned}$$

By the hypothesis on f, as $a \to 0$, $\varphi(a) \to \infty$ and $T(a) \to 0$; therefore the right-hand side converges to 0 as $a \to 0$. By the continuity of d and σ we get $d(\sigma(0, b), \sigma(0, b) \cdot \beta) = 0$. □

Theorem 5.3. *For any $b \in N$ and $\lambda \in \mathbb{L}$, $\sigma(0, b) \cdot \lambda = \sigma(0, b)$.*

Proof. It is proved in [5] that the action of the structure group on $\hat{P}$ is continuous. Therefore, the stabilizer S of $\sigma(0, b)$ is a closed subgroup; in fact, by the Cartan–von Neumann theorem, a Lie subgroup of $\mathbb{L}$. By Theorem 5.2, S contains the three 1-parameter subgroups $t \to \beta(t)$

induced by the boosts; hence the Lie algebra of S contains the three matrices $\dot{\beta}(0)$. However, it is known [6, III.2] that these three matrices span (as a Lie algebra) the Lie algebra of $\mathbb{L}$. By the bijective correspondence between connected Lie subgroups and Lie subalgebras, we get that the connected component of S is just $\mathbb{L}$, and $S = \mathbb{L}$. □

Let $c = (x, y) : [0, \tau) \to M$ be a maximal past null pregeodesic, y being a D-geodesic such that $y(0) = o$ and $\dot{y} = F_1 \circ y$. Consider the curve

$$C = (C_0, C_1, C_2, C_3) : [0, \tau) \to P$$

$$C_0 = \operatorname{ch}\theta(U \circ c) + \operatorname{sh}\theta(\dot{y}/f \circ x),$$

$$C_1 = \operatorname{sh}\theta(U \circ c) + \operatorname{ch}\theta(\dot{y}/f \circ x),$$

$$C_i = (1/f \circ x)P_i, \quad i = 2, 3,$$

where $\theta : [0, \tau) \to \mathbb{R}$ is given by $\exp(\theta) = \dot{x}/\dot{x}(0)$ and $P_i \in \mathcal{X}(y)$ is D-parallel such that $P_i(0) = F_i \circ y(0)$, $i = 2, 3$. By Lemmas 3.2 and 4.3, C is a horizontal curve. We give in (5.7) and (5.8) an alternative expression for C. First the cases $N = \mathbb{R}^3$ or $\mathbb{H}^3$. Since $P_i = F_i \circ y$ and $\dot{y}/(F \circ x) = E_1 \circ c$ we have

$$C(t) = (\sigma \circ c(t)) \cdot \beta(\theta(t)), \quad t \in [0, \tau) \ (N = \mathbb{R}^3, \mathbb{H}^3). \tag{5.7}$$

To answer the case $N = \mathbb{S}^3$, the reader will prove first an easy lemma.

Lemma 5.3. *Let X, Y be invariant fields in $\mathbb{S}^3$ such that $h(X, X) = 1, h(X, Y) = 0$, and let γ be an integral curve of X. Then*

$$P(t) = \cos(t)Y(\gamma(t)) + \sin(t)(Y \vee X)(\gamma(t))$$

is the parallel field along γ such that $P(0) = Y(\gamma(0))$. □

We get now that

$$C(t) = (\sigma \circ c(t)) \cdot \beta(\theta(t))\rho(t), \quad t \in [0, \tau) \ (N = \mathbb{S}^3). \tag{5.8}$$

Theorem 5.4. *Let $c = (x, y) : [0, \tau) \to M$ be a maximal null past pregeodesic, y being a unit-speed geodesic in N. If $C : [0, \tau) \to P$ is a horizontal lift of c such that $C_0(0) = U(c(0))$, we have that*

$$\lim_{t \to \tau} (\,\mathrm{d}(C(t), \sigma(0, y(t)))) = 0.$$

Proof. We claim that it is enough to deal with the case where $y(0) = o$. If $y^* : [0, \tau) \to N$ is any unit-speed D-geodesic, $y^*(0) = b$, we take the isometry φ of N as in Lemma 5.2 and then $\Phi(x, y) = (x, \varphi(y))$, which is an isometry of M. We define the geodesic y by

$Y^* = \varphi \circ y$ and c by $c^* = \Phi \circ c$. The lift $\Phi_\#$ of Φ to P, which is an isometry, verifies $C^* = \Phi_\# \circ C$ and

$$\begin{aligned} d(C^*(t), \sigma(0, y^*(t))) &= d(\Phi_\# \circ C(t), \sigma(0, \varphi \circ y(t))) \\ &= d(\Phi_\# \circ C(t), \Phi_\# \circ \sigma(0, y(t))) = d(C(t), \sigma(0, y(t))), \end{aligned}$$

proving our claim.

Suppose then, $y(0) = o$. We also say that we do not lose generality if we assume that $C(0) = \sigma(c(0))$. If this is not the case, we may choose $A \in SO(3)$ such that $C' = C \cdot A$ verifies the hypothesis and $C'(0) = \sigma(c(0))$. Since the right translation by A is an isometry of P and $\hat{P}$ we may write

$$\begin{aligned} d(C'(t), \sigma(0, y(t))) &= d(C(t) \cdot A, \sigma(0, y(t))) \\ &= d(C(t), \sigma(0, y(t) \cdot A^{-1})). \end{aligned}$$

By Theorem 5.3, $\sigma(0, y(t)) \cdot A^{-1} = \sigma(0, y(t))$; so if the left-hand side converges to 0 so does the right-hand side.

Let us tackle the case $C(0) = \sigma(c(0))$, which corresponds to C given by (5.7) or (5.8), and it is the only curve to be studied. For each $t \in [0, \tau)$ define

$$\gamma_t : [0, x(t)] \to M, \qquad \gamma_t(s) = (x(t) - s, y(t)).$$

(Notice that s is the parameter of γ_t, Γ_t and Δ_t soon to be defined.) Let Γ_t be its horizontal lift with initial condition $\Gamma_t(0) = \sigma(c(t))$. By Lemma 4.1 we have $\Gamma_t = \sigma \circ \gamma_t$. The curves (where β and ρ denote boosts and rotations)

$$\begin{aligned} &\Delta_t = \Gamma_t \cdot \beta(\theta(t)) \quad \text{if } N = \mathbb{R}^3, \mathbb{H}^3, \text{ and} \\ &\Delta_t = \Gamma_t \cdot \beta(\theta(t))\rho(t) \quad \text{if } N = \mathbb{S}^3, \end{aligned}$$

are also horizontal, with domain $[0, x(t)]$, and $\Delta_t(0) = C(t)$. To compute its length we write first

$$G(\dot{\Delta}_t, \dot{\Delta}_t)^{1/2} = \|\phi(\dot{\Delta}_t)\| = \|\beta(-\theta(t))(\phi(\dot{\Gamma}_t))\|,$$

because $\rho(t)$ preserves norms (ρ only appears if C is as in (5.8)). It is easy to check that $\phi(\dot{\Gamma}_t) = (-1, 0, \ldots, 0)$, hence $G(\dot{\Delta}_t, \dot{\Delta}_t)^{1/2} \leq 2\,ch(\theta(t))$. The inequalities $\dot{x} < 0$ and $\dot{f} > 0$ give $\ddot{x} = -(\dot{f} \circ x)\dot{x} > 0$ (see Theorem 2.3), thus $\dot{x}$ is increasing. By the definition of θ in (3.1)

$$G(\dot{\Delta}_t, \dot{\Delta}_t)^{1/2} \leq 2\frac{\dot{x}(t)}{\dot{x}(0)} = 2\frac{f(x(t))}{f(x(0))},$$

and

$$L(\Delta_t) \leq 2\frac{f(x(t))}{f(x(0))} x(t),$$

therefore $L(\Delta_t) \to 0$ as $t \to \tau$. The curve Δ_t starts at $C(t)$ and converges to $\sigma(0, y(t)) \cdot \lambda(t)$, $\lambda(t) = \beta(\theta(t))$ or $\beta(\theta(t))\rho(t)$. By Theorem 5.3, $\sigma(0, y(t)) \cdot \lambda(t) = \sigma(0, y(t))$, so $\mathrm{d}(C(t), \sigma(0, y(t)) \le L(\Delta_t) \to 0$. □

6. The singularities of the Friedmann model

We are interested in the maximal increasing solutions of the Friedmann equation such that $\lim_{t\to 0} f(t) = 0$. We get them for $C = \pm 1$ in the following lemmas.

Lemma 6.1. *Let $f : (0, l) \to \mathbb{R}$ be a maximal solution of $(\dot f^2 + 1)f = 2k$, $k > 0$, such that $0 < f < 2k$, $\dot f > 0$ and $\lim_{t\to 0} f(t) = 0$. Then $l = k\pi$ and the graph of f is parameterized by*

$$x(\lambda) = k(\lambda - \sin\lambda), \quad y(\lambda) = k(1 - \cos\lambda), \qquad \lambda \in (0, \pi),$$

that is, for each λ, $y(\lambda) = f(x(\lambda))$. □

Lemma 6.2. *Let $f : (0, l) \to \mathbb{R}$ be the maximal solution of $(\dot f^2 - 1)f = 2k$, $k > 0$, such that $f > 0$, $\dot f > 0$, and $\lim_{t\to 0} f(t) = 0$. Then the graph of f is parameterized by*

$$x(\lambda) = k(sh - \lambda), \quad y(\lambda) = k(ch\,\lambda - 1), \qquad \lambda > 0;$$

that is, for each $\lambda > 0$, $y(\lambda) = f(x(\lambda))$. □

We also have for $C = 0$ the equation $\dot f^2 f = 2k$. To get the simplest f we take $k = \frac{2}{9}$ and then $f(t) = t^{2/3}$, $t \in (0, \infty)$. In the cases $C = \pm 1$ we take $k = 1$. We will then study the Friedmann spaces $M = (0, l) \times N$ where $f : (0, l) \to \mathbb{R}$ is the *strictly increasing* function defined by

$$f : (0, \infty) \to \mathbb{R},\ f(t) = t^{2/3} \quad \text{if } N = \mathbb{R}^3, \tag{6.1a}$$

$$f : (0, \pi) \to \mathbb{R},\ f(\lambda - \sin\lambda) = 1 - \cos\lambda \quad \text{if } N = \mathbb{S}^3, \tag{6.1b}$$

$$f : (0, \infty) \to \mathbb{R},\ f(\mathrm{sh}\,\lambda - \lambda) = \mathrm{ch}\,\lambda - 1 \quad \text{if } N = \mathbb{H}^3. \tag{6.1c}$$

If $N = \mathbb{R}^3$ or $\mathbb{H}^3$, f gives the "full" Friedman model; if $N = \mathbb{S}^3$, f only gives the "left half" of the closed Friedmann model. These spaces will be studied in Theorems 6.1–6.3. We deal with the full closed model in Theorem 6.4.

Clearly, in the last two cases,

$$\lambda = \int_0^\lambda d\lambda = \int_0^\lambda \frac{\dot x(\lambda)\,\mathrm{d}\lambda}{y(\lambda)} = \int_0^\lambda \frac{\dot x(\lambda)\,\mathrm{d}\lambda}{f(x(\lambda))} = \int_0^{x(\lambda)} \frac{\mathrm{d}\xi}{f(\xi)}. \tag{6.2}$$

We define for the functions f,

$$G : [0, l) \to \mathbb{R}, \quad G(t) = \int_0^t \frac{\mathrm{d}\xi}{f(\xi)}.$$

Eq. (6.2) can be rewritten as $\lambda = G(x(\lambda))$ in cases (6.1b) and (6.1c). In case (6.1a) we have $G(t) = 3t^{1/3}$.

Lemma 6.3. *For all $\epsilon, s > 0$ there is a sequence $(a_1, \dots, a_k)$ in $I = (0, l)$, such that $\sum a_n < \epsilon$ and $\sum G(a_n) = s$.*

Proof. It is enough to show that there is an *infinite* sequence $\alpha_n, n \geq 1$, in I such that $\sum \alpha_n < \epsilon$, $\sum G(\alpha_n) > s$. Indeed, if it exists, define $a_0 = 0$ and let $k \geq 1$ be such that $\sum_{n<k} G(\alpha_n) < s \leq \sum_{n\leq k} G(\alpha_n)$. If we define $a_n = \alpha_n$ for $n < k$ and a_k by the condition $\sum_{n\leq k} G(a_k) = s$, the conclusion of the lemma holds.

To show the existence of the α_n for $f(x) = x^{2/3}$ we define $\alpha_n = (c/n)^3$, with a suitably chosen c to get $\sum \alpha_n < \epsilon$. (One uses essentially that $\sum(1/n)$ is divergent and $\sum(1/n^3)$ is convergent).

If f is as in Lemmas 6.1 or 6.2 we define $\alpha_n = x(c/n)$, where c a positive constant. The divergence of $\sum(1/n)$ and (6.2) give $\sum G(\alpha_n) = \infty$. We study $\sum \alpha_n$. If f is as in Lemma 6.1, note that $t - \sin(t)$ is an alternating series; therefore we may bound $t - \sin(t) \leq t^3/3!$ and

$$\sum \alpha_n = \sum x(c/n) \leq (c^3/6) \sum (1/n^3) < \epsilon$$

for a suitable choice of c. If f is as in Lemma 6.2, we have for $h = sh(1) - 1$ and $t \leq 1$ that

$$\mathrm{sh}\,(t) - t = \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = t^3 \left(\sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n+1)!} \right) \leq t^3 \left(\sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \right) = ht^3.$$

Therefore, for $c \leq 1$,

$$\sum \alpha_n = \sum x(c/n) \leq (c^3 h) \sum (1/n^3) < \epsilon$$

for a suitable choice of c. □

We use Theorem 2.3. For $a \in (0, l)$ let x_a be the maximal solution of $\dot{x} = -f(x)$ with $x_a(0) = a$; it is defined on the interval $[0, G(a))$. Let $y : \mathbb{R} \to N$ be a unit-speed geodesic with $y(0) = o = 0, 1, e_0$ and $y(s) = b, s > 0$. For each $r \in [0, s]$ we consider the geodesics $y_r^w(t) = y(r + wt)$, $w = \pm 1$. The curves

$$c_{a,r}^w = (x_a, y_r^w) : [0, G(a)) \to M, \quad w = \pm 1,$$

are the maximal null past pregeodesics starting at $(a, y_r^w(0)) = (a, y(r))$. Denote by $C_{a,r}^w$ its horizontal lifts such that $C_{a,r}^w(0) = \sigma(a, y(r))$. If we join these two curves we get a broken curve $Q_{(a,r)}$. By theorem 5.4 its endpoints are $\sigma(0, y_r^w(G(a))) = \sigma(0, y(r + wG(a)))$, $w = \pm 1$, and by Lemma 4.3,

$$\begin{aligned} &\mathrm{d}(\sigma(0, y^1(G(a))), \sigma(0, y^{-1}(G(a)))) \\ &\leq L(Q_{(a,r)}) \leq 2\sqrt{2} \int_0^a \frac{f(\xi)\, \mathrm{d}\xi}{f(a)} \leq 4a, \end{aligned} \tag{6.3}$$

because f is a strictly increasing function.

Theorem 6.1. *The set $\sigma(\{0\} \times N)$ has just one point.*

Proof. It is enough to show that for any $b \in N$ and $\epsilon > 0$ there exists a broken curve joining $\sigma(0, o)$ to $\sigma(0, b)$ with length less than ϵ. We may write $b = y(s)$, $s > 0$, for some unit-speed D-geodesic such that $y(0) = o$, and we choose a sequence a_n, $n = 1, \dots, k$ as in Lemma 6.3 such that

$$2\sum G(a_n) = s, \quad 4\sum a_n < \epsilon.$$

We define $a_0 = 0$ and $r_n = 2G(a_0) + \cdots + 2G(a_{n-1}) + G(a_n)$, $n = 1, \dots, k$. We have then k curves in the form $Q_{a,r}$, for $a = a_n$ and $r = r_n$; these curves will be denoted by Q_n. If we join the curves Q_n we have a broken curve with endpoints $\sigma(0, y(0))$ and $\sigma(0, y(2\sum G(a_n))) = \sigma(0, y(s))$; this curve has length $\leq 4\sum a_n < \epsilon$ according to (6.3).□

The only point in $\sigma(\{0\} \times N) \subseteq \hat{P} - P$ will be denoted by p and $s = \pi(p)$ will be called the essential past singularity.

It is well known [6] that any λ in $\mathbb{L}$ can be factorized as $\lambda = \rho\beta\sigma$, with $\rho, \sigma \in SO(3)$ and β a 1-boost. If r is the parameter of β, it may be characterized as the hyperbolic angle between e and $\lambda(e)$, $e = (1, 0, \dots, 0)$, so it is independent of the factorization.

Theorem 6.2. *For $(a, b) \in M$ and $\lambda \in \mathbb{L}$ with parameter r we have*

$$\mathrm{d}(\sigma(a, b) \cdot \lambda, p) \leq \sqrt{2}a \exp(-|r|).$$

Proof. Consider a factorization $\lambda = \rho\beta\sigma$ as above; let $s = \mathrm{sign}(r) = \pm 1$ and define $(v^1, v^2, v^3) \in \mathbb{R}^3$ by $\rho(-1, s, 0, 0) = (-1, v^1, v^2, v^3)$. Clearly $\sum (v^i)^2 = 1$, hence the vector $\xi = -U(a, b) + \sum v^i E_i(a, b)$ is a null vector. Let $c = (x, y) : [0, \tau) \to M$ be the maximal null past *geodesic* such that $c(0) = (a, b)$. Define C as its horizontal lift such that $C(0) = \sigma(a, b) = (U(a, b), E_i(a, b))$. The curve $K = C \cdot \lambda$ is also horizontal, and it is well known that the horizontal lift K of a geodesic verifies $\phi(\dot{K})$ is constant, ϕ being the fundamental form. Therefore,

$$\begin{aligned}\phi(\dot{K}) &= \phi(\dot{K}(0)) = \lambda^{-1}(\phi(\dot{C}(0)) = (\rho\beta\sigma)^{-1}(-1, v^1, v^2, v^3) \\ &= \sigma^{-1}\beta^{-1}(-1, -s, 0, 0).\end{aligned}$$

It is very easy to check that

$$\begin{pmatrix} \mathrm{ch}\,\alpha & \mathrm{sh}\,\alpha \\ \mathrm{sh}\,\alpha & \mathrm{ch}\,\alpha \end{pmatrix}\begin{pmatrix} 1 \\ s \end{pmatrix} = \exp(s\alpha)\begin{pmatrix} 1 \\ s \end{pmatrix} \quad \text{for } s = \pm 1.$$

It follows from this and formula (2.7) that

$$G(\dot{K}, \dot{K})^{1/2} = \|\phi(\dot{K}(0))\| = \sqrt{2}\exp(-sr),$$

$$L(K) = \sqrt{2}\exp(-sr)\tau = \sqrt{2}\exp(-sr)\int_0^a \frac{f(\xi)\,\mathrm{d}\xi}{f(a)} \leq \sqrt{2}\exp(-sr),$$

since f is an increasing function, Finally, K converges to $\sigma(0, y(\tau)) \cdot \lambda$ (see Theorem 5.4). By Theorem 6.1, this point is just p, hence

$$\mathrm{d}(\sigma(a,b)\cdot\lambda, p) \leq L(K) \leq \sqrt{2}a\exp(-sr).$$

Theorem 6.3. *Let (M, g) be the Friedmann space defined by f as in (6.1a)–(6.1c). The only neighborhood of the essential singularity s is the full b-completion $\bar{M}$; therefore any curve in M coverges to s.*

Proof. Let W be a neighborhood of s in $\bar{M}$ and $(a, b) \in M$. Clearly $\pi^{-1}(W)$ is a neighborhood of p, hence there is a ball B centered at p with radius ϵ such that $B \subseteq \pi^{-1}(W)$. For any $(a, b) \in M$, we may choose a boost β with parameter r such that, by Theorem 6.2, $d(\sigma(a,b)\cdot\lambda, p) \leq \sqrt{2}a\exp(-|r|) \leq \epsilon$. We have then that B intersects $\pi^{-1}(a, b)$, hence $(a, b) \in \pi(B) \subseteq W$. □

In the "full" Friedmann model, f is a maximal solution of the Friedmann equation $(\dot{f}^2 + 1)f = 2$, which is defined on $(0, 2\pi)$. Since f is symmetric with axis $x = \pi$, we can perform "symmetric" constructions getting two open submanifolds M_0 and M_1 given by $x < \pi$ and $x > \pi$ and $x > \pi$ with frame bundles P_0 and P_1, sections σ_0 and σ_1 and points $p_0 \in \bar{P}_0$ and $p_1 \in \bar{P}_1$ which are the image of $\sigma_0(\{0\} \times \mathbb{S}^3) \cdot \mathbb{L}$ and $\sigma_1(\{2\pi k\} \times \mathbb{S}^3 \cdot \mathbb{L}$.

Theorem 6.4. *With the preceeding notation, we have $p_0 = p_1$ in $\bar{P}$; therefore the essential singularities s_0 and s_1 are the same point s. This is the only singularity of the closed Friedmann model and the only neighborhood of s is the full b-completion $\bar{M}$.*

Proof. We show that for any $\epsilon > 0$ we have $d(p_0, p_1) < 3\epsilon$. Choose $r > 0$ such that $\sqrt{2}k\exp(-r) < \epsilon$. Let β be a boost with parameter r. The map $\sigma \cdot \beta : M \to P$ is continuous and there is $\delta > 0$ such that, for 1 the unit in $\mathbb{S}^3$,

$$d(\sigma(\pi k - \delta, 1)\cdot\beta, \sigma(\pi k + \delta, 1)\cdot\beta) < \epsilon.$$

We have then, for $q_\pm = \sigma(\pi k \pm \delta, 1)\cdot\beta$,

$$d(p_0, p_1) \leq d(p_0, q_-) + \mathrm{d}(q_-, q_+) + d(q_+, p_1) < 3\epsilon,$$

using once more Theorem 6.2.

Once we know that $p = p_1 = p_2$, we have, by application of Theorem 6.2 and its dual version to M_0 and M_1, the inequalities for $0 < a < \pi$,

$$d(\sigma(a,b)\cdot\lambda, p) \leq \sqrt{2}a\exp(-|r|), \qquad \square$$

$$d(\sigma(\pi + a, b)\cdot\lambda, p) \leq \sqrt{2}(\pi - a)\exp(-|r|).$$

By continuity,

$$d(\sigma(a,b)\cdot\lambda, p) \leq \sqrt{2}\pi\exp(-|r|) \quad \text{for all } (a,b) \in M = (0, 2\pi) \times \mathbb{S}^3. \tag{6.4}$$

With this, one may prove that s has just one neighborhood as in Theorem 6.3.

Let us finally prove that s is the only singularity. As we said in Section 1, all singularities are induced by b-incomplete curves so, we only need show that if $c = (x, y) : [0, 1) \to M$ is a b-incomplete curve there is a horizontal lift C of c converging to p. Since $L(C) < \infty$, it is enough to show that for some sequence $t_n \to 1$, $C(t_n) \to p$. The reason is that for any pair of sequences s_n and s'_n with limit 1, $C(s_n)$ and $C(s'_n)$ are equivalent Cauchy sequences.

Choose a sequence $t_n \to 1$ such that $c(t_n) \to z \in [0, 2\pi] \times \mathbb{S}^3$. Write $C = (\sigma \circ c) \cdot \lambda$ for $\lambda : [0, 1) \to \mathbb{L}$ and let r_n be the parameter of $\lambda(t_n)$. Choosing subsequences if necessary we may assume that either (a) $|r_n| \to \infty$, or (b) $|r_n| \leq H$ for all n. If (a) holds, (6.4) implies that

$$\mathrm{d}(C(t_n), p) \leq 2\sqrt{2}\pi \exp(-|r_n|) \to 0.$$

The section $\sigma = (E_0, E_1, E_2, E_3) = (0, 2\pi) \times \mathbb{S}^3 \to P$ can be continuously extended to a map, still denoted by the same symbol, $\sigma : [0, 2\pi] \times \mathbb{S}^3 \to \hat{P}$, such that $\sigma(a, b) = p$ if $a = 0, 2\pi$. By the continuity of σ, $(\sigma \circ c)(t_n) \to p \in \hat{p}$. If (b) holds, by compactness, taking again subsequences if necessary, we may assume that $\lambda_n \to \lambda \in \mathbb{L}$. Therefore, $C(t_n)$ converges to $p \cdot \lambda$, which is just p by Theorem 5.3 and its (unwritten) "symmetric" version. □

We return to the nonclosed Friedmann model. Is there another singularity besides the essential singularity s? We cannot answer. As we said in Section 1, all singularities are obtainable as projections of endpoints of horizontal lifts of b-incomplete curves. We show at least that future causal geodesics are *not* b-incomplete. At any rate it seems difficult to get a physical interpretation for singularities $s' \neq s$.

Lemma 6.4. *Let $c = (x, y) : [0, \tau) \to M$ be a maximal future causal geodesic. Then x is not bounded.*

Proof. By causality, $-\dot{x}^2 + (f \circ x)^2 h(\dot{y}, \dot{y}) \leq 0$; hence conditions $\dot{x}, f > 0$ give

$$h(\dot{y}, \dot{y})^{1/2} \leq \frac{\dot{x}}{f \circ x},$$

$$L_h(y) \leq \lim_{t \to \tau} \int_0^t \frac{\dot{x}(u)\,\mathrm{d}u}{f \circ x(u)} = \lim_{t \to \tau} \int_{x(0)}^{x(t)} \frac{\mathrm{d}\xi}{f(\xi)}.$$

If the increasing function x were bounded, it would have a limit; then $L_h(y) < \infty$ and y would converge too for N is complete. Then c is not maximal. □

Theorem 6.5. *Maximal future causal geodesics in the nonclosed Friedmann spaces considered are not b-incomplete for its horizontal lifts have infinite length.*

Proof. Let c be an integral curve of U; i.e. $c(t) = (a + t, b)$, $0 < t < \infty$. We may prove as we did in Lemma 4.1 that the curve $C = (C_0, \ldots, C_3)$ given by

$$C_0 = U \circ c, \quad C_i(t) = \frac{f(a)}{f(a + t)} \xi_i,$$

where $\xi_i \in T_bN$ and $h(\xi_i, \xi_j) = f(a)^{-2}\delta_{ij}$, is a horizontal lift of c and $G(\dot{C}, \dot{C}) = 1$. Since C is defined on a nonbounded interval, $L(C) = \infty$.

Suppose now that $c = (x, y) : [0, \tau) \to M$ is a causal future geodesic, but not an integral curve of U. Then $\dot{y}(t) \neq 0$ for all t and we may reparameterize y to a unit-speed curve γ; $y = \gamma \circ r$. By (2.3b), γ is actually a unit-speed geodesic. Since length and horizontal lifts are invariant under reparameterization, we only need to show that if $c = (x, y) : [0, \tau) \to M$ is a causal future *pregeodesic* there is a horizontal lift C with infinite length. We use that $x(t) \to \infty$ as $t \to \tau$ (Lemma 6.4) and that y is a unit-speed geodesic.

We define C with the help of Lemma 3.2. Indeed, by suitable choices of X_0, Y_0 and P we have that the fields along c

$$\begin{aligned}
C_0 &= \mathrm{ch}\,\theta(U \circ c) + \mathrm{sh}\,(\theta)(\dot{y}/f \circ x),\\
C_1 &= \mathrm{sh}\,\theta(U \circ c) + \mathrm{ch}\,(\theta)(\dot{y}/f \circ x),\\
C_i &= P_i/(f \circ x), \quad i = 2, 3, \quad P_i \in \mathcal{X}(y),\\
h(\dot{y}, P_i) &= 0 \ \text{ and } \ DP_i = 0
\end{aligned}$$

are parallel provided that $\dot{\theta} + \dot{f} \circ x = 0$. We choose θ with $\theta(0) = 0$; then, $\dot{f}$ being positive, we have $\theta \leq 0$. We easily get

$$\begin{aligned}
U \circ c &= \mathrm{ch}\,\theta C_0 - \mathrm{sh}\,\theta C_1, \quad \dot{y}/(f \circ x) = -\mathrm{sh}\,\theta C_0 + \mathrm{ch}\,\theta C_1;\\
\dot{c} &= \dot{x}(U \circ c) + \dot{y} = [\dot{x}\mathrm{ch}\,\theta - (f \circ x)\mathrm{sh}\,\theta]C_0 + [-\dot{x}\mathrm{sh}\,\theta + (f \circ x)\mathrm{ch}\,\theta]C_1;\\
G(\dot{C}, \dot{C}) &= [\dot{x}\mathrm{ch}\,\theta - (f \circ x)\mathrm{sh}\,\theta]^2 + [-\dot{x}\mathrm{sh}\,\theta + (f \circ x)\mathrm{ch}\,\theta]^2\\
&= (\dot{x}^2 + (f \circ x)^2\mathrm{ch}\,(2\theta) - 2\dot{x}(f \circ x)\mathrm{sh}\,(2\theta).
\end{aligned}$$

After the substitution $\theta = -|\theta|$ and $\mathrm{ch}\,(2|\theta|) \geq sh(2|\theta|)$ we obtain

$$G(\dot{C}, \dot{C}) \geq (\dot{x} + f \circ x)^2\mathrm{sh}\,(2|\theta|).$$

Clearly $(\mathrm{d}/\mathrm{d}t)|\theta| = -\dot{\theta} = \dot{f} \circ x > 0$; then there are numbers $r, S > 0$ such that if $t > r$, $\mathrm{sh}\,(|2\theta(t)|) > S$. We arrive at

$$L(C) \geq L(C|_{[r,\tau]}) \geq S \int_r^\tau \dot{x}(t)\,\mathrm{d}t = S(x(\tau) - x(t)).$$

If $L(C)$ were finite, x would be bounded. Contradiction. □

Acknowledgements

The first author was supported by Proyecto de Investigación Complutense number PR156/97-17104.

References

[1] B. Bosshard, On the b-boundary of the closed Friedmann model, Commun. Math. Phys. 46 (1976) 263–268.
[2] D. Canarutto, An introduction to the geometry of singularities in general relatively, Riv. Nuovo Cimento 11 (3) (1988) 1–60.
[3] C.T.J. Dodson, Spacetime edge geometry, Int. J. Theoret. Phys. 17 (6) (1978) 389–504.
[4] S.W. Hawking, G.F.R. Ellis, The Large-Scale Structure of Spacetime, University Press, Cambridge, 1973.
[5] R.A. Johnson, The bundle boundary in some special cases, J. Math. Phys. 18 (5) (1977) 898-902.
[6] M.A. Naimark, Les représentations linéaires du groupe de Lorentz, Dunod, Paris, 1962.
[7] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1982.
[8] B.G. Schmidt, A new definition of singular points in general relativity, Gen. Rel. Grav. 1 (3) (1971) 269–280.